

# Q-Deformed Anisotropic Superexchange Interaction, Frustration and $GL_{pq}(2)$

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## Abstract

We study a suitable  $q$ -deformed version of the Moriya's superexchange interaction theory by means of its underlying quantum group structure. We show that the one-dimensional chain case is associated with the non-standard quantum group  $GL_{pq}(2)$ , evidencing the integrability structure of the system. This biparametric deformation of  $GL(2, \mathbf{C})$  arise as a twisting of  $GL_q(2)$  and it match exactly the local rotation appearing in the Shekhtman's work [1]. This allow us to express the frustration condition in terms of this twisting, also showing that effect of the Moriya's vector amounts to a twisting of the boundary condition.

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The presence of weak ferromagnetism in various antiferromagnetic compounds was explained at the end of 50's by Dzyaloshinskii [2], from purely symmetry grounds, introducing in the thermodynamic potential of those systems an antisymmetric spin-spin interaction term  $\mathbf{D} \cdot (\mathbf{M}_1 \times \mathbf{M}_2)$ , where  $\mathbf{M}_1$  and  $\mathbf{M}_2$  are sublattices magnetization, and  $\mathbf{D}$  a macroscopic parameter called Dzyaloshinskii's vector. The microscopic basis for the Dzyaloshinskii conjecture was given by Moriya [3]. He extended the Anderson's superexchange interaction theory [4] in order to include in the one-electron Hamiltonian an spin-orbit coupling. In terms of electron annihilators and creations operators,  $c$  and  $c^\dagger$ , the resulting Hamiltonian is (considering here just one orbital state per ion, denoted in [3] by  $n$ )  $H = H_o + H_t$ , being:

$$H_o = \sum_r \sum_\alpha \varepsilon_r c_{r\alpha}^\dagger c_{r\alpha}$$

$$H_t = \sum_{r,r'} \sum_{\alpha,\alpha'} t_{rr'} c_{r\alpha}^\dagger \left( e^{i\theta_{rr'}} \hat{\mathbf{d}}_{rr'} \cdot \boldsymbol{\sigma} \right)_{\alpha,\alpha'} c_{r'\alpha'}$$

where  $r, r'$  runs over lattice sites,  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  are the Pauli matrices (so  $\alpha, \alpha' = \{\uparrow, \downarrow\}$ ),  $\varepsilon_r$  are crystal energies, and  $t_{rr'}, \theta_{rr'}, \hat{\mathbf{d}}_{rr'}$  come from the transfer integrals  $b_{rr'} = b_{r'r}^*$  and  $\mathbf{C}_{rr'} = \mathbf{C}_{r'r}^*$  [3], which in the case of *non degenerate orbital ground state* (except for being a Kramer's doublet)  $b_{rr'}$  is real and  $\mathbf{C}_{rr'}$  is purely imaginary and we can write  $b_{rr'} = t_{rr'} \cos \theta_{rr'}$  and  $\mathbf{C}_{rr'} = i t_{rr'} \sin \theta_{rr'} \hat{\mathbf{d}}_{rr'}$  with  $t_{rr'}, \theta_{rr'} \in \mathbf{R}$  and  $\hat{\mathbf{d}}_{r'r} \in \mathbf{R}^3$  ( $\hat{\mathbf{d}}_{r'r} \cdot \hat{\mathbf{d}}_{r'r} = 1$ ). The  $\theta_{rr'} = 0$  case corresponds to the Anderson's theory, while  $\theta_{rr'} \neq 0$  give rise to an anisotropy term as consequence of the spin-orbit interaction. From now on we omit, for simplicity, the  $r$ 's indices in all parameters. Following [3] and [4], and taking into account the correct factor 4 pointed out in [1], we arrive to an effective Hamiltonian, up to a constant term:

$$\mathcal{H} = \sum_{\langle r, r' \rangle} \frac{t^2}{U} \{ \cos 2\theta (\boldsymbol{\sigma}_r \cdot \boldsymbol{\sigma}_{r'} - \hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_r \hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_{r'}) +$$

$$+ \hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_r \hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_{r'} + \mathbf{I} + \sin 2\theta \hat{\mathbf{d}} \cdot (\boldsymbol{\sigma}_r \times \boldsymbol{\sigma}_{r'}) \}$$

where  $\langle r, r' \rangle$  runs now over nearest neighbor,  $\boldsymbol{\sigma}_r$  are Pauli matrices acting on the  $r$ -th site<sup>1</sup>,  $\mathbf{I}$  the identity operator and  $U$  is the so-called Hubbard energy (to put two electrons on the same

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<sup>1</sup>In general, given a collection of isomorphics linear spaces  $\{V_i\}$  and an operator  $\mathcal{O}$  that acts on

ion). The Hamiltonian  $\mathcal{H}$  represents the *anisotropic superexchange interaction (ASI)*, where the last term is called Dzyaloshynskii-Moriya (DM) interaction (the microscopic counterpart of the Dzyaloshynskii interaction), and  $t^2/U \sin 2\theta \hat{\mathbf{d}}$  is the Moriya's vector for the bond  $r, r'$ <sup>2</sup>. For  $\theta = 0$ ,  $\mathcal{H} = H_{xxx}$ , i.e., the  $XXX$  or isotropic Heisenberg model.

At this point Shekhtman *et al.* [1] showed that, term by term,  $\mathcal{H}$  can be cast in an isotropic Heisenberg model form, i.e.:

$$\mathcal{H}^{r,r'} = \frac{t^2}{U} \{ \sigma_r \cdot \sigma'_{r'} + \mathbf{I} \}$$

with  $\sigma'_{r'} = e^{-i\theta \hat{\mathbf{d}} \cdot \sigma_{r'}} \sigma_{r'} e^{i\theta \hat{\mathbf{d}} \cdot \sigma_{r'}}$ , i.e.,  $\sigma_{r'}$  rotated in  $-2\theta$ <sup>3</sup>. This local rotations, defined bond to bond, are similarity transformations term by term, i.e.  $\mathcal{H}^{r,r'} = e^{-i\theta \hat{\mathbf{d}} \cdot \sigma_{r'}} H_{xxx}^{r,r'} e^{i\theta \hat{\mathbf{d}} \cdot \sigma_{r'}}$ , but not in general for the whole Hamiltonians. A sufficient condition to extend the local rotations to a similarity transformation on the whole lattice is that the product of such rotations along any closed path on the lattice be equal to  $(-1)^n \mathbf{I}$  ( $n \in \mathbf{Z}$ ) (this condition is just a compatibility requirement of the local transformations to make this extension). In the 1-D open chains this condition always holds (there is no closed paths), but for closed ones it depends on the boundary. If this condition holds it means that there is *no frustration* (or the transformation is *non frustrated*), and the Hamiltonians are similar, hence  $\mathcal{H}$  and  $H_{xxx}$  have the same eigenspectra, being the similarity transformation in 1-D closed chain

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$\otimes_{i=1}^M V_i$ , we will denote  $\mathcal{O}_{i_1 \dots i_M}$  as the operator that acts non trivially on  $\otimes_{k=1}^M V_{i_k}$  and as the identity in the rest.

<sup>2</sup>This antisymmetric coupling has been observed ten years ago in the high-temperature superconducting material  $\text{La}_2\text{CuO}_4$  [5], and was studied on some cuprates in [6]. Then, this mechanism was used to describe several properties in doped  $\text{La}_2\text{CuO}_4$  as  $\text{La}_{2-x}\text{Ba}_x \text{CuO}_4$  [7]. Recently Affleck *et al.* [8] have used the DM interaction to explain the field-induced gap in antiferromagnetic (quasi-one-dimensional) Cu Benzoate chains.

<sup>3</sup>Actually, in Shekhtman's work it makes a rotation in  $\theta$  for  $r$  and  $-\theta$  for  $r'$ . It can see the relevant is the relative angle between the rotations. This angle is just  $-2\theta$ .

$$\mathbf{A} = \prod_{r=2}^N \prod_{j=1}^{r-1} e^{-i\theta_{r-j} \hat{\mathbf{d}}_{r-j} \cdot \boldsymbol{\sigma}_r} \quad (1)$$

with  $N$  the number of sites, and denoting  $\tau_{r,r'} = \tau_{r,r+1} \equiv \tau_r$  for all parameters  $\tau$ . So they conclude that the *frustration is a necessary condition to have weak ferromagnetism* (because the  $XXX$ 's ground state have zero magnetization -is purely antiferromagnetic-).

The main aim of our work is to show up the quantum group structure associated to this model and study the frustration conditions in terms of its algebraic content. To this end we are going to make a suitable generalization of the model allowing us to reveal these underlying algebraic structure, which is deeply tied up to the integrability of the model. We generalize  $H$  by making an analytic continuation through  $\theta \rightarrow \theta_c$ , such that writing  $\theta_c = \theta + i \phi$  ( $\alpha, \beta \in \mathbf{R}$ ) and defining

$$\begin{aligned} p &= e^{2i\theta} \\ q &= e^{-2\phi} \end{aligned} \quad (2)$$

$H_t$  becomes in:

$$H_t = \sum_{\langle r, r' \rangle} \sum_{\alpha, \alpha'} t \, c_{r\alpha}^\dagger \left( p^{\hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_r / 2} q^{\hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_{r'} / 2} \right)_{\alpha, \alpha'} c_{r'\alpha'}$$

and  $\mathcal{H}$ , up to a constant term:

$$\begin{aligned} \mathcal{H}_q &= \sum_{\langle r, r' \rangle} \frac{t^2}{U} \left\{ \frac{p + p^{-1}}{2} (\boldsymbol{\sigma}_r \cdot \boldsymbol{\sigma}_{r'} - \hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_r \hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_{r'}) \right. \\ &\quad \left. + \frac{q + q^{-1}}{2} (\hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_r \hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_{r'} + \mathbf{I}) + \frac{p - p^{-1}}{2i} \hat{\mathbf{d}} \cdot (\boldsymbol{\sigma}_r \times \boldsymbol{\sigma}_{r'}) \right\} \end{aligned} \quad (3)$$

and we name it  $q$ -deformed  $ASI$  ( $q - ASI$ ). We can see that  $p = 1 \Rightarrow \mathcal{H}_q = H_{xxz}$ , i.e., the  $XXZ$  or anisotropic Heisenberg model (we actually have an inhomogeneous version of the  $XXZ$ , unless  $\hat{\mathbf{d}}$  and  $q$  were the same for all bonds, so we can make an appropriate global rotation to pass from  $\hat{\mathbf{d}}$  to  $\hat{\mathbf{z}}$  and have the proper  $XXZ$ ).

Let us analyze the effects of the frustration for the  $q$ -deformed Hamiltonian. As for  $q = 1$ , we can write term by term  $\mathcal{H}_q^{r,r'} = p^{-\hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_{r'} / 2} H_{xxz}^{r,r'} p^{\hat{\mathbf{d}} \cdot \boldsymbol{\sigma}_{r'} / 2}$ . In this case, the non frustration condition is not enough to extend this rotations to a similarity transformation between  $\mathcal{H}_q$

and  $H_{xxz}$ , the Moriya's directions  $\{\hat{\mathbf{d}}_{r,r'}\}$  being all equals is needed, i.e.,  $\hat{\mathbf{d}}_{r,r'} = \hat{\mathbf{d}} \forall r, r'$  (the nature of this fact is the  $SU(2)$ -invariance of  $H_{xxx}$ , while  $H_{xxz}$  is just  $U(1) \otimes \mathbf{Z}_2$ -invariant). In the 1-D case, the similarity transformation is given by putting in  $\mathbf{A}$  (see ec. (1))  $\hat{\mathbf{d}}_r = \hat{\mathbf{d}} \forall r$ . Then, for the  $q$ -*ASI* model, when  $\hat{\mathbf{d}}_{r,r'} = \hat{\mathbf{d}} \forall r, r'$ , the previous conclusion still holds, because if there is no frustration  $\mathcal{H}_q$  is similar to  $H_{xxz}$  which have a purely antiferromagnetic ground state and there is no net ferromagnetic moment (remember  $q = e^{-2\phi}$ ).

Let's briefly review how to construct Hamiltonian integrable models from a given  $\mathbf{R}(x)$  solution of the quantum Yang-Baxter equation (QYBE)

$$\mathbf{R}_{12}(x/y) \mathbf{R}_{13}(x) \mathbf{R}_{23}(y) = \mathbf{R}_{23}(y) \mathbf{R}_{13}(x) \mathbf{R}_{12}(x/y)$$

with  $\mathbf{R}(x)$  acting on  $V_a \otimes V$ ,  $V_a$  and  $V$  being isomorphic linear spaces. Regarding this  $\mathbf{R}(x)$ -matrix as a representation of the Lax operators of a quantum spin chain or as the Boltzman weights of some statistical model [9] [10], one built up the monodromy matrix (see also [11] and references therein)  $\mathbf{T}_a(x) = \mathbf{R}_{a,1}(x) \cdot \mathbf{R}_{a,2}(x) \cdots \mathbf{R}_{a,N}(x)$  giving rise to transfer matrix:

$$\mathbf{t}(x) = \text{tr}_a [\mathbf{T}_a(x)]$$

where the trace is over the auxiliary  $V_a$  space tied to each site, so  $\mathbf{t}^{(N)}$  is an operator acting on  $V^{\otimes N}$ . These monodromy matrix can be encoded into a bialgebra structure, the Yang-Baxter algebra (YBA), which in some limit of the spectral parameter becomes quasitriangular [12]. In this framework, the transfer matrices  $\{\mathbf{t}(x)\}$ , with different spectral parameter  $x$ , appears as a set of commuting quantities from which one derive an associated Hamiltonian for the system

$$\mathbf{H} = c \frac{d}{dx} \ln [\mathbf{t}(x)]_{x=x_o}$$

thus leading to the integrability of the system [10] ( $x_o$  is some appropriate value of  $x$  and  $c \in \mathbf{C}$ ). The locality of the Hamiltonian is warranted by choosing (if there exist)  $x_o$  such that  $\mathbf{R}(x_o) = \alpha \mathbf{P}$ , with  $\mathbf{P}$  the permutation matrix, then  $\mathbf{H} = \sum_{k=1}^N H_{k,k+1}$  with  $H_{k,k+1} = c/\alpha \frac{d}{dx} \mathbf{R}_{k,k+1}(x_o) \mathbf{P}_{k,k+1}$  acting non trivially only in spaces  $k$ -th and  $(k+1)$ -th of  $V^{\otimes N}$ , and

$H_{N,N+1} \equiv H_{N,1}$ . For later convenience, we introduce in  $\mathbf{R}_{a,k}(x)$  a transformation which preserves integrability  $\mathbf{R}_{a,k} \rightarrow \mathbf{\Gamma}_k^{(k)} \mathbf{R}_{a,k} \left( \mathbf{\Gamma}_k^{(k)} \right)^{-1}$ , and defining  $\mathbf{S} = \prod_{k=1}^N \mathbf{\Gamma}_k^{(k)}$  it drives to a Hamiltonian  $\mathbf{H}^{\mathbf{S}} = \mathbf{S} \cdot \sum_{k=1}^N H_{k,k+1} \cdot \mathbf{S}^{-1}$ .

It is well known that the quantum deformation of  $GL(2, \mathbf{C})$ , namely  $GL_q(2)$ , is the underlying algebraic structure of the  $XXZ$  model (see [9], and references therein), with associated  $R$ -matrix:

$$\mathbf{R}^q = \begin{bmatrix} q & \cdot & \cdot & \cdot \\ \cdot & 1 & q - q^{-1} & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & q \end{bmatrix}$$

In fact, defining  $\mathbf{R}^q(x) = x \mathbf{R}^q - \frac{1}{x} \mathbf{P} \cdot (\mathbf{R}^q)^{-1} \cdot \mathbf{P}$  (with its associated YBA), we built up, for  $x = 1$ , the homogeneous Hamiltonian  $\mathbf{H}_q^{\mathbf{S}} = \mathbf{S} \cdot \sum_{k=1}^N H_{k,k+1}^q \cdot \mathbf{S}^{-1}$ , or explicitly

$$\begin{aligned} \mathbf{H}_q^{\mathbf{S}} = \frac{c}{q - q^{-1}} \sum_{k=1}^N \{ & \sigma_k \cdot \sigma_{k+1} - \hat{\mathbf{d}} \cdot \sigma_k \hat{\mathbf{d}} \cdot \sigma_{k+1} \\ & + \frac{q + q^{-1}}{2} (\hat{\mathbf{d}} \cdot \sigma_k \hat{\mathbf{d}} \cdot \sigma_{k+1} + \mathbf{I}) \} \end{aligned}$$

where if  $\hat{\mathbf{d}} = (\cos\gamma \ \sin\varphi, \sin\gamma \ \sin\varphi, \cos\varphi)$ <sup>4</sup> so  $\mathbf{S} = \exp[i\varphi \hat{\mathbf{u}} \cdot \sum_{r=1}^N \sigma_r/2]$  and  $\hat{\mathbf{u}} = (-\sin\gamma, \cos\gamma)$ . From ec. (3) for  $p = 1$ , it sees that  $\mathbf{H}_q^{\mathbf{S}} = H_{xxz}$  if  $c = t^2/U (q - q^{-1})$ .

The question naturally arise what is the underlying algebraic structure of the  $q$ - $ASI$ ? Introducing a 2-cocycle twisting transformation  $\Phi$  on  $GL_q(2)$  [13] [14], which maps the algebraic structure into another equivalent now characterized by the  $R$ -matrix  $\mathbf{R}^{pq}$ , given by:

$$\mathbf{R}^q \rightarrow \mathbf{R}^{pq} = \tilde{\Phi} \mathbf{R}^q \Phi^{-1}$$

being  $\Phi = \rho \otimes \mathbf{I}$  with  $\rho = p^{-\hat{\mathbf{z}} \cdot \sigma/2}$ ,  $\tilde{\Phi} = \Phi_{21} = \mathbf{P} \cdot \Phi \cdot \mathbf{P} = \mathbf{I} \otimes \rho$ ; which is associated to the non-standard quantum group  $GL_{pq}(2)$  [15]. Proceeding as in  $\mathbf{R}^q$  case described above, we

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<sup>4</sup>It's totally equivalent to work with  $\hat{\mathbf{z}}$  however, in order to preserve the connection with the Moriya's vector concept, we prefer take an arbitrary direction.

introduce  $\mathbf{R}^{pq}(x) = x \mathbf{R}^{pq} - \frac{1}{x} \mathbf{P} \cdot (\mathbf{R}^{pq})^{-1} \cdot \mathbf{P}$  and build up the homogeneous Hamiltonian  $\mathbf{H}_{pq}^{\mathbf{S}}$ :

$$\begin{aligned} \mathbf{H}_{pq}^{\mathbf{S}} = & \frac{c}{q - q^{-1}} \sum_{k=1}^N \left\{ \frac{p + p^{-1}}{2} (\sigma_k \cdot \sigma_{k+1} - \hat{\mathbf{d}} \cdot \sigma_k \hat{\mathbf{d}} \cdot \sigma_{k+1}) + \right. \\ & + \frac{q + q^{-1}}{2} (\hat{\mathbf{d}} \cdot \sigma_k \hat{\mathbf{d}} \cdot \sigma_{k+1} + \mathbf{I}) \\ & \left. + \frac{p - p^{-1}}{2i} \hat{\mathbf{d}} \cdot (\sigma_k \times \sigma_{k+1}) \right\} \end{aligned} \quad (4)$$

It is worth remarking that, since the construction, the monodromy matrix  $\mathbf{T}(x)$  of this system satisfy a quadratic algebra relation (YBA),

$$\mathbf{R}^{pq}(x/y) \mathbf{T}(x) \otimes \mathbf{T}(y) = \mathbf{T}(y) \otimes \mathbf{T}(x) \mathbf{R}^{pq}(x/y)$$

from which one derives the integrability of the system.

In this way, we found an integrable spin chain associated to the quantum group  $GL_{pq}(2)$  that strongly resembles  $\mathcal{H}_q$ . In fact, in the particular case where  $p$  and  $q$  are those of (2) and  $c = t^2/U (q - q^{-1})$ ,  $\mathbf{H}_{pq}^{\mathbf{S}} = \mathcal{H}_q$  (see ec. (3)) for the homogeneous periodic chain case. This tell us that the quantum group  $GL_{pq}(2)$  is the underlying algebraic structure of the  $q$ -*ASI* model, allowing us to understood it as twisted version of *XXZ* model. There is a nice connection of the twisting  $\Phi$  with the map found in [1] as we shall explain below.

Let's work out the effect of the twisting on the Hamiltonian. Building up a Hamiltonian  $\mathbf{H}^{\mathbf{S}} = \sum_{k=1}^N H_{k,k+1}^{\mathbf{S}} = \mathbf{S} \cdot \sum_{k=1}^N H_{k,k+1} \cdot \mathbf{S}^{-1}$ , from an  $\mathbf{R}$ -matrix such that  $\mathbf{R}(x=1) \propto \mathbf{P}$ , then from  $\tilde{\Phi} \mathbf{R} \Phi^{-1}$  the Hamiltonian is

$$\mathbf{H}_{\Phi}^{\mathbf{S}} = \sum_{k=1}^N \tilde{\Phi}_{k,k+1}^{\mathbf{S}} \cdot H_{k,k+1}^{\mathbf{S}} \cdot (\tilde{\Phi}^{\mathbf{S}})^{-1}_{k,k+1} \quad (5)$$

In our case  $\mathbf{H}^{\mathbf{S}} = H_{xxz}$ ,  $\mathbf{H}_{\Phi}^{\mathbf{S}} = \mathcal{H}_q$  and  $\tilde{\Phi}_{k,k+1}^{\mathbf{S}} = \mathbf{S} \cdot \tilde{\Phi}_{k,k+1} \cdot \mathbf{S}^{-1} = p^{-\hat{\mathbf{d}} \cdot \sigma_{k+1}/2}$ . So, the consequences of the twist on the Hamiltonian boils down exactly to the rotation pointed out in Shekhtman's work [1], and the necessary and sufficient condition to avoid frustration is that

$$p = e^{i2n\pi/N}, \quad n \in \mathbf{Z}$$

In other words, if  $p = e^{i2n\pi/N}$  ( $n \in \mathbf{Z}$  and  $N \in \mathbf{N}$ ), then there is no frustration if and only if we built an  $N$ -sites Hamiltonian. Then, although the twisting leads to integrable spin chains for arbitrary values of the parameter  $p$ , the *no frustration* constraint requires  $p$  being a root of the unit and it is just in this case when the twisting becomes in a similarity transformation between  $H_{xxz}$  and  $\mathcal{H}_q$ .

As we could guess for 1-D closed chains, the relevant information about the effect of the Moriya's vector can be put it in terms of boundary conditions. In fact, noting that  $[H_{k,k+1}, \rho_k \rho_{k+1}] = 0$  (remember  $\tilde{\Phi}_{k,k+1} = \rho_{k+1}$ ), we can write  $\mathbf{S} = \hat{\mathbf{S}} \cdot \mathbf{X}$  with  $\mathbf{X} = \prod_{l=2}^N (\rho_l)^{l-1}$ , taking  $\mathbf{H}_{\Phi}^{\mathbf{S}}$  the form:

$$\mathbf{H}_{\Phi}^{\mathbf{S}} = \hat{\mathbf{S}} \cdot \left[ \sum_{k=1}^N H_{k,k+1} + (\Omega_1)^{-1} H_{N,1} \Omega_1 \right] \cdot \hat{\mathbf{S}}^{-1} \quad (6)$$

with  $\Omega = \rho^N$ . So  $\mathcal{H}_q$  is always similar to  $H_{xxz}^{tbc}$ , i.e., the  $XXZ$  with twisted boundary conditions ( $tbc$ ) given by  $\Omega$ . In this terms, i.e. as an  $XXZ$  model with  $tbc$ , the Hamiltonian  $\mathbf{H}_{\Phi}^{\mathbf{S}}$  have just been widely studied by means of the Bethe equations [16] [17]. Again, when there is no frustration, i.e.  $\Omega = (-1)^n \mathbf{I}$ , one recovers a similarity with the standard  $H_{xxz}$ .

Most of the systems what manifest a DM interaction have  $p$ 's that change bond to bond, like the so-called canonical DM antiferromagnet that alternate  $p$  and  $1/p$  on successive bonds. So to have Hamiltonians that describe this systems we shall introduce inhomogeneities in  $\mathbf{H}_{pq}^{\mathbf{S}}$ . It is possible to introduce some kind of inhomogeneity, preserving integrability, by means the transformation  $\mathbf{R}(x) \rightarrow \widehat{\mathbf{R}}^{(k)}(x) = \mathbf{R}(x) (\Psi^{(k)} \otimes \mathbf{I})$ , where  $\Psi^{(k)} \in \mathcal{G}_{\mathbf{R}}$ , being  $\mathcal{G}_{\mathbf{R}} = \{\Psi\}$  the group of operators satisfying  $[\mathbf{R}(x), \Psi \otimes \Psi] = 0 \ \forall x$ , and representing the *internal symmetries* of the YBA associated to  $\mathbf{R}(x)$  (see [18] and references therein). The Hamiltonian we built with  $\widehat{\mathbf{R}}$ 's is

$$\widehat{\mathbf{H}}^{\mathbf{S}} = \mathbf{S} \cdot \sum_{k=1}^N \left( \Psi_{k+1}^{(k)} \right)^{-1} H_{k,k+1} \Psi_{k+1}^{(k)} \cdot \mathbf{S}^{-1} \quad (7)$$

How in  $\widehat{\mathbf{H}}^{\mathbf{S}}$  appear  $\Psi$  and  $\Psi^{-1}$  the group we can consider is the quotient  $\widehat{\mathcal{G}}_{\mathbf{R}} = \mathcal{G}_{\mathbf{R}} / \mathcal{Z}$ , where  $\mathcal{Z}$  is the group of scalar matrices on  $V_a$ . It is straightforward to show that for  $\mathbf{R}^{pq}$ :



$$\hat{\mathcal{G}}_{\mathbf{R}^{pq}} = \begin{cases} \{\mu^{-\hat{\mathbf{z}} \cdot \sigma/2}\} \text{ if } p \neq 1 \\ \{\mu^{-\hat{\mathbf{z}} \cdot \sigma/2}\} \cup \{\xi^{-\hat{\mathbf{x}} \cdot \sigma/2}\} \text{ if } p = 1 \end{cases}$$

$\mu, \xi \in \mathbf{C} - \{0\}$ , and calling  $[\hat{\mathcal{G}}_{\mathbf{R}^{pq}}]_{p \neq 1} = \{\Psi^\mu\}_{\mu \in \mathbf{C}}$ , from ecs. (5) and (7) we see that the change in  $\mathbf{H}_{pq}^{\mathbf{S}}$  is equivalent to put  $p\mu_k$  instead of  $p$  in (4). In such a case there is frustration *iff*

$$\prod_{k=1}^N [p\mu_k] \neq e^{i2n\pi/N}$$

$\forall n \in \mathbf{Z}$ . Choosing  $\mu_{2k} = 1$  and  $\mu_{2k+1} = p^{-2}$  we have the canonical case <sup>5</sup>.

Because of  $[\mathbf{R}(x), \Psi \otimes \Psi] = 0 \ \forall x$  implies that  $[H_{k,k+1}, \Psi_k \Psi_{k+1}] = 0$ , the  $\Psi$ 's can be cumulated in the last term of  $\hat{\mathbf{H}}^{\mathbf{S}}$ , as above (ec. (6)), through  $\mathbf{X} = \prod_{l=2}^N \prod_{k=1}^{l-1} \Psi_l^{(l-k)}$ , having  $\Omega = \prod_{k=1}^N \Psi^{(N+1-k)}$ . So the associated Hamiltonian is, up to a similarity transformation, the original one with twisted boundary conditions. In our case, the twisted boundary conditions will be given by  $\Omega = \prod_{k=1}^N [p\mu_k]^{-\hat{\mathbf{z}} \cdot \sigma/2}$ . Going back, it sees that it can associate this Hamiltonian to  $GL_{p'q}(2)$ , with  $p'$  any  $N$ -root  $\left\{ \prod_{k=1}^N [p\mu_k] \right\}^{1/N}$ , instead of  $GL_{pq}(2)$ .

When  $q = 1$ , the Hamiltonian  $[\mathbf{H}_{pq}^{\mathbf{S}}]_{q=1} \equiv \mathbf{H}_p^{\mathbf{S}}$  can be made completely inhomogeneous by using again  $\{\Psi^\mu\}_{\mu \in \mathbf{C}}$  and a similarity transformation  $\mathbf{Y} = \prod_{k=2}^N \Lambda_k^{(k)}$  with

$$\Lambda_k^{(k)} = \left[ \prod_{j=1}^{k-1} [p\mu_{k-j}]^{-\hat{\mathbf{d}}_{k-j} \cdot \sigma_k/2} \right] \cdot \left\{ \prod_{j=1}^N [p\mu_j] \right\}^{(k-1) \hat{\mathbf{d}} \cdot \sigma_k/2}$$

changing  $(p, \hat{\mathbf{d}}) \rightarrow \{p\mu_k, \hat{\mathbf{d}}_k\}_{k=1}^N$ . So we have a Hamiltonian  $\hat{\mathbf{H}}_p^{\mathbf{Y} \cdot \mathbf{S}} = \mathcal{H}$  for 1-D closed chains, hence  $\mathcal{H}$  is similar to  $H_{xxx}^{tbc}$  (instead of  $H_{xxz}^{tbc}$ ). The frustration condition and the boundary's twisting  $\Omega$  are, of course, the same of the case above.

It is important to note that we can construct  $\mathbf{H}_{pq}^{\mathbf{S}}$  from  $GL_q(2)$  and  $\hat{\mathcal{G}}_{\mathbf{R}^q} \supset \hat{\mathcal{G}}_{\mathbf{R}^{pq}}$  (using  $\Psi^{(k)} = p^{-\hat{\mathbf{d}} \cdot \sigma/2}$  for all  $k$ ), but we want to encode the Moriya's vectors content on a proper algebraic structure instead on its internal symmetries.

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<sup>5</sup>In such a case there is no frustration for all  $p$  *iff*  $N$  is even.

Concluding, we have shown that the integrability of the DM interaction rest on the underlying quantum group structure  $GL_{pq}(2)$ , connecting the 2-cocycle twisting  $\Phi$  with the rotation that maps the Hamiltonian in a  $H_{xxz}$  like one. The existence of this map is well understood in terms of the no frustration of the twisting along the whole chain.

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